MATH 1010E University Mathematics
Lecture Notes (week 8)
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1 L'Hospital's Rule

Another useful application of mean value theorems is *L'Hospital's Rule*. It helps us to evaluate limits of "*indeterminate forms*" such as $\frac{0}{0}$. Let's look at the following example. Recall that we have proved in week 3 (using the sandwich theorem and a geometric argument)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

We say that the limit above has indeterminate form $\frac{0}{0}$ since both the numerator and denominator goes to 0 as $x \to 0$. Roughly speaking, L'Hospital's rule says that under such situation, we can differentiate the numerator and denominator first and then take the limit. The result, if exists, should be equal to the original limit. For example,

$$\lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \frac{\cos x}{1} = 1,$$

which is equal to the limit before we differentiate!

Theorem 1.1 (L'Hospital's Rule) *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions in (a, b) and fix an $x_0 \in (a, b)$. Assume that*

(i) $f(x_0) = 0 = g(x_0)$.

(ii) $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$ *(i.e. the limit exists and is finite).*

Then, we have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L.$$

Example 1.2 *Consider the limit*

$$\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x},$$

this is a limit of indeterminate form $\frac{0}{0}$. Therefore, we can apply L'Hospital's Rule to obtain

$$\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x} = \lim_{x \to 0} \frac{(\sin^2 x)'}{(1 - \cos x)'},$$

if the limit on the right hand side exists. Since the right hand side is the same as

$$\lim_{x\to 0}\frac{2\sin x\cos x}{\sin x}=\lim_{x\to 0}(2\cos x)=2.$$

Therefore, we conclude that $\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}=2.$

Exercise: Calculate the limit in Example 1.2 without using L'Hospital's Rule (hint: $\sin^2 x = 1-\cos^2 x$).

Sometimes we have to apply L'Hospital's Rule a few times before we can evaluate the limit directly. This is illustrated by the following two examples.

Example 1.3 *Consider the limit*

$$\lim_{x\to 0}\frac{x-\sin x}{x^3},$$

this is of the form "$\frac{0}{0}$*". Therefore, by L'Hospital's rule*

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2},$$

if the right hand side exists. The right hand side is still in the form "$\frac{0}{0}$*", therefore we can apply L'Hospital's Rule again*

$$\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x},$$

if the right hand side exists. But now the right hand side can be evaluated:

$$\lim_{x\to 0}\frac{\sin x}{6x}=\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{6}.$$

As a result, if we trace backwards, we conclude that the original limit exists and

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{1}{6}.$$

Example 1.4 *Consider the limit*

$$\lim_{x\to 0}\frac{e^x-x-1}{1-\cosh x}.$$

Applying L'Hospital's Rule twice, we can argue as in Example 1.3 that

$$\lim_{x\to 0}\frac{e^x-x-1}{1-\cosh x}=\lim_{x\to 0}\frac{e^x-1}{-\sinh x}=\lim_{x\to 0}\frac{e^x}{-\cosh x}=\frac{1}{-1}=-1.$$

After seeing these examples, let us now go back to give a proof of L'Hospital's Rule.

Proof of L'Hospital's Rule: Recall Cauchy's Mean Value Theorem which says that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

for some $\xi \in (a,b)$. Therefore, since $f(x_0) = g(x_0) = 0$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(\xi)}{g'(\xi)}$$

for some ξ between x and x_0. Notice that as $x \to x_0$, we must also have $\xi \to x_0$. Therefore, we have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{\xi \to x_0} \frac{f'(\xi)}{g'(\xi)}.$$

This proves the L'Hospital's Rule.

2 Other indeterminate forms

When we evaluate limits, there are other possible "indeterminate forms", for example

$$\frac{0}{0}, \qquad 0\cdot\infty, \qquad \frac{\infty}{\infty}, \qquad 0^0. \tag{2.1}$$

Note that these forms above are just *formal* expressions which does not have very precise mathematical meanings as ∞ is not a real number.

Convention: We distinguish two "infinities" by writing

$$\infty := +\infty \qquad \text{and} \qquad -\infty := -\infty.$$

Remark 2.1 *Not all expressions involving* 0 *and* ∞ *would result in an indeterminate form. For example,*

$$0^\infty = 0, \qquad \infty^\infty = \infty, \qquad \infty + \infty = \infty, \qquad \infty\cdot\infty = \infty.$$

In this section, we will see that all the indeterminate forms in (2.1) can actually be rewritten into the standard form $\frac{0}{0}$. Symbolically we have $1/0 = \infty$. Therefore,

$$0\cdot\infty = 0\cdot\frac{1}{0} = \frac{0}{0},$$

$$\frac{\infty}{\infty} = \frac{1/0}{1/0} = \frac{0}{0}.,$$

$$0^0 = \exp(0 \ln 0) = \exp(0 \cdot (-\infty)) = \exp(-\frac{0}{0}).$$

We should emphasize that the "calculations" above are just formal. They indicate the general idea of transforming the limits rather than actual arithmetic of numbers. Using these ideas, we can actually handle all the determinate forms in (2.1) by the L'Hospital's Rule. We have

Theorem 2.2 (L'Hospital's Rule) *The same conclusion holds if we replace (i) by*

$$\lim_{x \to x_0} f(x) = \pm\infty = \lim_{x \to x_0} g(x).$$

Remark 2.3 *The theorem also holds in the case $x_0 = \pm\infty$ and for one-sided limits as well.*

We postpone the proof of Theorem 2.2 until the end of this section but we will first look at a few applications.

Example 2.4 *Consider the one-side limit*

$$\lim_{x \to 0^+} x \ln x.$$

This is of the form $0 \cdot (-\infty)$. However, we can rewrite it as

$$x \ln x = \frac{\ln x}{1/x},$$

which is of the form $\frac{-\infty}{\infty}$ as $x \to 0^+$. Therefore, we can apply Theorem 2.2 to conclude that

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

Therefore, we have $\lim_{x \to 0^+} x \ln x = 0$. In words, this means that as $x \to 0^+$, the linear function x is going to 0 faster than the logarithm function $\ln x$ going to $-\infty$.

Example 2.5 *Sometimes we have to apply L'Hospital's Rule a few times. For example,*

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0.$$

Similarly, we can prove that

$$\lim_{x\to+\infty}\frac{x^k}{e^x}=0, \qquad \text{for any } k.$$

In other words, as $x\to+\infty$, the exponential function e^x is going to ∞ faster than any polynomial of x.

The following example shows that L'Hospital's Rule may not always work:

$$\lim_{x\to\infty}\frac{\sinh x}{\cosh x}=\lim_{x\to\infty}\frac{\cosh x}{\sinh x}=\lim_{x\to\infty}\frac{\sinh x}{\cosh x},$$

which gets back to the original limit we want to evaluate! So L'Hospital's Rule leads us nowhere in such situation. For this example, we have to do some cancellations first,

$$\lim_{x\to\infty}\frac{\sinh x}{\cosh x}=\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}}=1.$$

3 Some tricky examples of L'Hospital's Rule

Sometimes it is not very obvious how we should transform a limit into a "standard" indeterminate form.

Example 3.1 *Evaluate that limit*

$$\lim_{x\to\infty}x\sin\frac{1}{x}.$$

We can choose to transform it to either

$$x\sin\frac{1}{x}=\frac{\sin(1/x)}{1/x} \qquad \text{or} \qquad x\sin\frac{1}{x}=\frac{x}{1/\sin(1/x)}.$$

The first one has the form "$\frac{0}{0}$" and the second one has the form "$\frac{\infty}{\infty}$" as $x\to\infty$. Therefore, we can apply L'Hospital's Rule to both cases. For the first case, we have

$$\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{x\to\infty}\frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\cos\frac{1}{x}=1.$$

However, for the second case, we have

$$\lim_{x\to\infty}\frac{x}{1/\sin(1/x)}=\lim_{x\to\infty}\frac{1}{\frac{1}{x^2}\frac{\cos(1/x)}{\sin^2(1/x)}},$$

which doesn't seem to simplify after L'Hospital's Rule. Therefore, sometimes we have to choose a good way to transform the limit before we apply the L'Hospital's Rule. A general rule of thumb here is that the expression should get simpler after taking the derivatives.

Example 3.2 *Evaluate the limit*

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right).$$

This limit has the indeterminate form "$\infty - \infty$", which we haven't mentioned. There is in fact no general way to evaluate limits of such forms. But for this particular example, we can transform it as

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x},$$

which has the standard indeterminate form "$\frac{0}{0}$". Therefore, we can apply L'Hospital's Rule a few times to get

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=0.$$

Example 3.3 *Evaluate the limit*

$$\lim_{x\to\infty} x^{\frac{1}{x}}.$$

Recall that if $a>0, b$ are *real* numbers, we define $a^b := \exp(b\ln a)$. Therefore,

$$\lim_{x\to\infty} x^{\frac{1}{x}}=\lim_{x\to\infty}\exp\left(\frac{1}{x}\ln x\right)=\exp\left(\lim_{x\to\infty}\frac{\ln x}{x}\right)=\exp\left(\lim_{x\to\infty}\frac{1/x}{1}\right)=e^0=1.$$

Note that we can move the limit into the function "exp" since the exponential function "exp" is continuous.

We end this section with a proof of Theorem 2.2.

Proof of Theorem 2.2: The idea is that if $f(x_0)=\pm\infty=g(x_0)$, then we have $\frac{1}{f(x_0)}=0=\frac{1}{g(x_0)}$. Therefore, we can apply L'Hospital's Rule to conclude that

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{1/g(x)}{1/f(x)}=\lim_{x\to x_0}\frac{-g'(x)/g(x)^2}{-f'(x)/f(x)^2}.$$

The right hand side is

$$\lim_{x\to x_0}\left(\frac{g'(x)}{f'(x)}\cdot\frac{f(x)^2}{g(x)^2}\right)=\lim_{x\to x_0}\frac{g'(x)}{f'(x)}\cdot\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right)^2=\lim_{x\to x_0}\frac{g'(x)}{f'(x)}\cdot\left(\lim_{x\to x_0}\frac{f(x)}{g(x)}\right)^2.$$

Therefore, we have

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{g'(x)}{f'(x)}\cdot\left(\lim_{x\to x_0}\frac{f(x)}{g(x)}\right)^2.$$

Canceling and moving terms around, we obtain

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\left(\lim_{x\to x_0}\frac{g'(x)}{f'(x)}\right)^{-1}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

This proves the theorem.

Question: Spot the gaps in the above proof. Can you fix them?

4 Indefinite Integral

Now, we know how to differentiate a function $f(x)$ to get a new function $f'(x)$. We want to ask whether we can reverse the process.

Question: Given a function $f:(a,b)\to\mathbb{R}$ (say differentiable), can we find another function $F:(a,b)\to\mathbb{R}$ such that

$$F'(x)=f(x)$$

for all $x\in(a,b)$?

Let's try to understand the above question by some simple examples.

Example 4.1 *Suppose $f(x)=e^x$. Can we solve for $F(x)$ such that $F(x)=f(x)=e^x$? Well we know that*

$$F(x)=e^x$$

is a solution since $(e^x)'=e^x$. Are there any other solutions? Yes, for example,

$$F(x)=e^x+1$$

is another solution. In fact, for any constant $C\in\mathbb{R}$,

$$F(x)=e^x+C$$

is a solution. Are these all the solutions then? The answer is indeed YES!

Question: Show that if there are two solutions $F_1(x)$ and $F_2(x)$ such that $F_1'(x) = f(x) = F_2'(x)$ then

$$F_1(x) = F_2(x) + C \qquad \text{for some constant } C.$$

We make the following definition.

Definition 4.2 *If $F(x)$ is a differentiable function such that $F'(x) = f(x)$, we say that $F(x)$ is a* primitive function *of $f(x)$ and*

$$\int f(x)\, dx := F(x) + C$$

is said to be the indefinite integral *of $f(x)$. Here, $C \in \mathbb{R}$ is an arbitrary constant called the* integration constant*.*

For example, since we know that $(e^x)' = e^x$ and $x' = 1$,

$$\int e^x\, dx = e^x + C,$$

$$\int 1\, dx = x + C.$$

Proposition 4.3 *We can evaluate some of the elementary indefinite integrals.*

1. $\displaystyle\int \cos x\, dx = \sin x + C$.
2. $\displaystyle\int \sin x\, dx = -\cos x + C$.
3. $\displaystyle\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$ *for any* real *number $n \neq -1$.*
4. $\displaystyle\int \frac{1}{x}\, dx = \ln|x| + C$.

The following property helps us evaluate a much larger class of indefinite integrals.

Proposition 4.4 (Linearity) *Indefinite integrals are linear:*

1. $\displaystyle\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$.

2. $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx$ *for any constant* k*.*

Using just Proposition 4.3 and 4.4, we can evaluate a lot of indefinite integrals. First, we know how to compute the indefinite integrals of any polynomials. For example,

$$\begin{aligned}
\int (x^4 - 3x + 7)\,dx &= \int x^4\,dx - 3\int x\,dx + 7\int 1\,dx \\
&= \left(\frac{x^5}{5} + C_1\right) - 3\left(\frac{x^2}{2} + C_2\right) + 7(x + C_3) \\
&= \frac{x^5}{5} - \frac{3x^2}{2} + 7x + (C_1 - 3C_2 + 7C_3) \\
&= \frac{x^5}{5} - \frac{3x^2}{2} + 7x + C
\end{aligned}$$

where C is ANY constant. Note that in the end, we can simply group all the constants together to form a single constant C since all these constants are arbitrary.

We can also evaluate the indefinite integrals of some rational functions. For example,

$$\begin{aligned}
\int \frac{(x+2)^2}{x}\,dx &= \int \frac{x^2 + 4x + 4}{x}\,dx \\
&= \int \left(x + 4 + \frac{4}{x}\right)\,dx \\
&= \int x\,dx + \int 4\,dx + 4\int \frac{1}{x}\,dx \\
&= \frac{x^2}{2} + 4x + 4\ln|x| + C.
\end{aligned}$$

The powers in the rational function do not need to be integers in some cases.

$$\begin{aligned}
\int \frac{5x^2 + \sqrt{x} + 3}{\sqrt{x}}\,dx &= \int (5x^{\frac{3}{2}} + 1 + 3x^{-\frac{1}{2}})\,dx \\
&= 5\frac{x^{5/2}}{5/2} + x + 3\frac{x^{1/2}}{1/2} + C \\
&= 2x^{\frac{5}{2}} + x + 6x^{\frac{1}{2}} + C.
\end{aligned}$$

Proposition 4.5 *We can evaluate the following indefinite integrals of trigonometric functions.*

1. $\displaystyle\int \sec^2 x \, dx = \tan x + C.$

2. $\displaystyle\int \csc^2 x \, dx = -\cot x + C.$

3. $\displaystyle\int \sec x \tan x \, dx = \sec x + C.$

4. $\displaystyle\int \csc x \cot x \, dx = -\csc x + C.$

5. $\displaystyle\int \sec x \, dx = \ln|\sec x + \tan x| + C.$

6. $\displaystyle\int \csc x \, dx = -\ln|\csc x + \cot x| + C.$

The first four integrals are easy. The last two will be proved after we learn t-substitution in the next class.